EFFECTIVE BIRATIONALITY OF PLURICANONICAL SYSTEMS

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Abstract

By using the theory of AZD originated by the author, I prove that for every smooth projective n-fold X of general type and every

$$m \geq \lceil \sum_{\ell=1}^{n} \sqrt[\ell]{2}\ \ell \rceil + 1,$$

$\mid mK_X \mid$ gives a birational rational map from X into a projective space, unless it has a nontrivial (relative dimension is positive) rational fiber space structure whose general fiber is birational to a variety of relatively low degree in a projective space. MSC 32J25

Contents

1 Introduction

Let X be a smooth projective variety and let K_X be the canonical bundle of X. X is said to be of general type, if K_X is big, i.e.,

$$\limsup_{m\to\infty} m^{-\dim X}\dim H^0(X,\mathcal{O}_X(mK_X)) > 0$$

holds. The following problem is fundamental to study projective vareity of general type.

Problem Let X be a smooth projective variety of general type. Find a positive integer m_0 such that for every $m \geq m_0$, $\mid mK_X \mid$ gives a birational rational map from X into a projective space.

If $\dim X = 1$, it is well known that $\mid 3K_X \mid$ gives a projective embedding. In the case of smooth projective surfaces of general type, E. Bombieri showed that $\mid 5K_X \mid$ gives a birational rational map from X into a projective space ([5]). In the case of $\dim X \geq 3$, I have proved the following theorem.

Theorem 1.1 *([20]) There exists a positive integer ν_n which depends only on n such that for every smooth projective n-fold X of general type defined over complex numbers, $\mid mK_X \mid$ gives a birational rational map from X into a projective space for every $m \geq \nu_n$.*

Theorem 1.1 is an affirmative answer to the problem. But it seems to be very hard to give an **effective estimate** of the number ν_n because the proof depends on the abstract facts of Hilbert scheme.

The main purpose of this article is to give the following **weak effective answer** to the problem.

Theorem 1.2 *For every smooth projective n-fold X of general type, one of the followings holds.*

1. *for every*

$$m \geq \lceil \sum_{\ell=1}^{n} \sqrt[\ell]{2}\ \ell \rceil + 1,$$

$\mid mK_X \mid$ *gives a birational rational map from X into a projective space,*

2. *X is dominated by a family of subvarieties of dimension $d(\geq 1)$ which are birational to subvarieties of degree less than or equal to $(\lceil\sum_{\ell=1}^{n}\sqrt[\ell]{2}\ \ell\rceil + 1)^d$ in a projective space by some pluricanonical system $\mid \alpha K_X \mid$.*

Theorem 1.3 *For every smooth projective n-fold X of general type, one of the followings holds.*

1. *for every*

$$m \geq \lceil\sum_{\ell=1}^{n}\sqrt[\ell]{2}\ \ell\rceil + 1,$$

$\mid mK_X \mid$ gives a birational rational map from X into a projective space,

2. *there exists a rational fibration*

$$f : X - \cdots \to Y$$

such that a general fiber F of f is positive dimensional and is birational to a subvariety of degree less than or equal to $(\lceil\sum_{\ell=1}^{n}\sqrt[\ell]{2}\ \ell\rceil + 1)^{d^2}d^d$ in a projective space by some pluricanonical system $\mid \alpha K_X \mid$, where d denotes the dimension of F.

The proofs of Theorem 1.2 and 1.3 are technically much easier than that of Theorem 1.1 ([20]). But they are effective and clarify the essential obstruction to obtain the birationality of the pluricanonical map $\Phi_{|mK_X|}$ with relatively small m. As one see in the proof, if we need very large m to embed X birationally into a projective space by $\mid mK_X \mid$, the image $\Phi_{|mK_X|}(X)$ is distorted in the sense that X is small in the fiber direction and large in the horizontal direction with respect to a rational fiber space structure. Such a phenomenon was first observed by E. Bombieri in his paper [5]. Actually he found the existence of a genus 2 fibration is an obstruction to the birationality of $\mid 2K_X \mid$ for some surfaces of general type ([5, p. 173, Main Theorem (iv)]). Of course the results above will not be optimal and more abstract in comparison with the case of surfaces.

In the case of 3-folds of general type, there were several results[10, 11, 2] in this direction. But these results depend on the plurigenus formula for canonical 3-folds of general type and moreover their estimates depend on the apriori bound of $\chi(X, \mathcal{O}_X)$, hence it is even weaker than Theorem 1.1 in this respect and the bound is is so huge that they have only a theoretical interests.

I hope the estimates in Theorem 1.2 and Theorem 1.3 are acceptable at least for projective varieties of low dimension. And it seems to be more or less optimal in order of size, even in the case of arbitrary dimension. But I should say that even in the case of 3-folds, the exceptional cases seem to be very hard to classify.

As in [20], the main difficulty is the fact that K_X is not ample in general. To overcome this difficulty we use a special singular hemitian metric on K_X called AZD which was originated by the author ([16]). By using AZD we can handle K_X as if K_X were nef and big.

2 Preliminaries

2.1 Multiplier ideal sheaves

In this section, we shall review the basic definitions and properties of multiplier ideal sheaves.

Definition 2.1 *Let L be a line bundle on a complex manifold M. A singular hermitian metric h is given by*

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞-hermitian metric on L and $\varphi \in L^1_{loc}(M)$ is an arbitrary function on M.

The curvature current Θ_h of the singular hermitian line bundle (L, h) is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where $\partial\bar{\partial}$ is taken in the sense of a current. The L^2-sheaf $\mathcal{L}^2(L, h)$ of the singular hermitian line bundle (L, h) is defined by

$$\mathcal{L}^2(L, h) := \{\sigma \in \Gamma(U, \mathcal{O}_M(L)) \mid \ h(\sigma, \sigma) \in L^1_{loc}(U)\},$$

where U runs opens subsets of M. In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that

$$\mathcal{L}^2(L, h) = \mathcal{O}_M(L) \otimes \mathcal{I}(h)$$

holds. We call $\mathcal{I}(h)$ **the multiplier ideal sheaf of** (L, h). If we write h as

$$h = e^{-\varphi} \cdot h_0,$$

where h_0 is a C^∞ hermitian metric on L and $\varphi \in L^1_{loc}(M)$ is the weight function, we see that

$$\mathcal{I}(h) := \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi})$$

holds. We also denote $\mathcal{L}^2(\mathcal{O}_M, e^{-\varphi})$ by $\mathcal{I}(\varphi)$. Let (L, h) be a singular hermitian line bundle on a smooth projective variety X such that

$$\Theta_h \geq -\omega$$

holds for some C^∞ Kähler form ω on X. Then by [12, p. 561], we see that $\mathcal{I}(h)$ is a coherent sheaf of $\mathcal{O}_X$-ideal.

Similarly we obtain the sheaf

$$\mathcal{I}_\infty(h) := \mathcal{L}^\infty(\mathcal{O}_M, e^{-\varphi})$$

and call it **the L^∞-multiplier ideal sheaf** of (L, h). We have the following vanishing theorem.

Theorem 2.1 *(Nadel's vanishing theorem [12, p.561]) Let (L, h) be a singular hermitian line bundle on a compact Kähler manifold M and let ω be a Kähler form on M. Suppose that Θ_h is strictly positive, i.e., there exists a positive constant ε such that*

$$\Theta_h \geq \varepsilon\omega$$

holds. Then $\mathcal{I}(h)$ is a coherent sheaf of $\mathcal{O}_M$-ideal and for every $q \geq 1$

$$H^q(M, \mathcal{O}_M(K_M + L) \otimes \mathcal{I}(h)) = 0$$

holds.

2.2 Analytic Zariski decomposition

To study a big line bundle we introduce the notion of analytic Zariski decompositions. By using analytic Zariski decompositions, we can handle big line bundles like a nef and big line bundles.

Definition 2.2 *Let M be a compact complex manifold and let L be a line bundle on M. A singular hermitian metric h on L is said to be an analytic Zariski decomposition, if the followings hold.*

1. *Θ_h is a closed positive current,*

2. for every $m \geq 0$, the natural inclusion

$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \to H^0(M, \mathcal{O}_M(mL))$$

is isomorphim.

Remark 2.1 *If an AZD exists on a line bundle L on a smooth projective variety M, L is pseudoeffective by the condition 1 above.*

Theorem 2.2 *([16, 18] see also [20, Section 2.2]) Let L be a big line bundle on a smooth projective variety M. Then L has an AZD.*

2.3 Volume of subvarieties

To measure the positivity of big line bundles on a projective variety we shall introduce a volume of a projective variety with respect to a line bundle.

Definition 2.3 *Let L be a line bundle on a compact complex manifold M of dimension n. We define the L-volume of M by*

$$\mu(M, L) := n! \cdot \overline{\lim}_{m\to\infty} m^{-n} \dim H^0(M, \mathcal{O}_M(mL)).$$

Definition 2.4 *([19]) Let (L, h) be a singular hermitian line bundle on a smooth projective variety X such that $\Theta_h \geq 0$. Let Y be a subvariety of X of dimension r. We define the volume $\mu(Y, L)$ of Y with respect to L by*

$$\mu(Y, L) := r! \cdot \overline{\lim}_{m\to\infty} m^{-r} \dim H^0(Y, \mathcal{O}_Y(mL) \otimes \mathcal{I}(h^m)/tor),$$

where tor denotes the torsion part of the sheaf $\mathcal{O}_Y(mL) \otimes \mathcal{I}(h^m)$.

3 Stratification of varieties by multiplier ideal sheaves

In this section we shall construct a stratification of a smooth projective variety X of general type by using an AZD h of K_X. We use the ideas in [1, 21] to construct the stratification. But since (K_X, h) is not an ample line bundle, the argument is a little bit more involved.

3.1 Construction of a stratification

Let X be a smooth projective n-fold of general type. Let h be an AZD of K_X. Let us denote $\mu(X, K_X)$ by μ_0. We set

$$X^\circ = \{x \in X \mid x \not\in \mathrm{Bs} \mid mK_X \mid \text{and } \Phi_{|mK_X|} \text{ is a biholomorphism}$$
$$\text{on a neighbourhood of } x \text{ for some } m \geq 1\}.$$

Then X° is a nonempty Zariski open subset of X.

Lemma 3.1 *Let x, x' be distinct points on X°. We set*

$$\mathcal{M}_{x,x'} = \mathcal{M}_x \otimes \mathcal{M}_{x'}$$

Let ε be a sufficiently small positive number. Then

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}}\rceil}) \neq 0$$

for every sufficiently large m, where $\mathcal{M}_x, \mathcal{M}_{x'}$ denote the maximal ideal sheaf of the points x, x' respectively.

Proof of Lemma 3.1. Let us consider the exact sequence:

$$0 \to H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}}\rceil}) \to H^0(X, \mathcal{O}_X(mK_X)) \to$$
$$H^0(X, \mathcal{O}_X(mK_X)/\mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}}\rceil}).$$

Since

$$n! \cdot \overline{\lim}_{m\to\infty} m^{-n} \dim H^0(X, \mathcal{O}_X(mK_X)/\mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m}{\sqrt[n]{2}}\rceil}) = \mu_0(1-\varepsilon)^n < \mu_0$$

hold, we see that Lemma 3.1 holds. **Q.E.D.**

Let us take a sufficiently large positive integer m_0 and let σ be a general (nonzero) element of $H^0(X, \mathcal{O}_X(m_0K_X) \otimes \mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m_0}{\sqrt[n]{2}}\rceil})$. We define a singular hermitian metric h_0 on K_X by

$$h_0(\tau, \tau) := \frac{\mid \tau \mid^2}{\mid \sigma \mid^{2/m_0}}.$$

Then

$$\Theta_{h_0} = \frac{2\pi}{m_0}(\sigma)$$

holds, where (σ) denotes the closed positive current defined by the divisor (σ). Hence Θ_{h_0} is a closed positive current. Let α be a positive number and let $\mathcal{I}(\alpha)$ denote the multiplier ideal sheaf of h_0^α, i.e.,

$$\mathcal{I}(\alpha) = \mathcal{L}^2(\mathcal{O}_X, (\frac{h_0}{h_X})^\alpha),$$

where h_X is an arbitrary C^∞-hermitian metric on K_X. Let us define a positive number $\alpha_0(= \alpha_0(x, y))$ by

$$\alpha_0 := \inf\{\alpha > 0 \mid (\mathcal{O}_X/\mathcal{I}(\alpha))_x \neq 0 \,\text{and}\, (\mathcal{O}_X/\mathcal{I}(\alpha))_{x'} \neq 0\}.$$

Since $(\sum_{i=1}^n \mid z_i \mid^2)^{-n}$ is not locally integrable around $O \in \mathbf{C}^n$, by the construction of h_0, we see that

$$\alpha_0 \leq \frac{n\sqrt[n]{2}}{\sqrt[n]{\mu_0}(1-\varepsilon)}$$

holds. Then one of the following two cases occurs.

Case 1.1: For every small positive number δ, $\mathcal{O}_X/\mathcal{I}(\alpha_0 - \delta)$ has 0-stalk at both x and x'.
Case 1.2: For every small positive number δ, $\mathcal{O}_X/\mathcal{I}(\alpha_0 - \delta)$ has nonzero-stalk at one of x or x' say x'.

We first consider Case 1.1. Let δ be a sufficiently small positive number and let V_1 be the germ of subscheme at x defined by the ideal sheaf $\mathcal{I}(\alpha_0 + \delta)$. By the coherence of $\mathcal{I}(\alpha)(\alpha > 0)$, we see that if we take δ sufficiently small, then V_1 is independent of δ. It is also easy to verify that V_1 is reduced if we take δ sufficiently small. In fact if we take a log resolution of $(X, \frac{\alpha_0}{m_0}(\sigma))$, V_1 is the image of the divisor with discrepancy -1 (for example cf. [7, p.207]). Let X_1 be a subvariety of X which defines a branch of V_1 at x. We consider the following two cases.

Case 2.1: X_1 passes through both x and x',
Case 2.2: Otherwise

For the first we consider Case 2.1. Suppose that X_1 is not isolated at x. Let n_1 denote the dimension of X_1. Let us define the volume μ_1 of X_1 with respect to K_X by

$$\mu_1 := \mu(X_1, K_X).$$

Since $x \in X^\circ$, we see that $\mu_1 > 0$ holds.

Lemma 3.2 *Let ε be a sufficiently small positive number and let x_1, x_2 be distinct regular points on $X_1 \cap X^\circ$. Then for a sufficiently large $m > 1$,*

$$H^0(X_1, \mathcal{O}_{X_1}(mK_X) \otimes \mathcal{I}(h^m) \otimes \mathcal{M}_{x_1,x_2}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon)\frac{m}{\sqrt[n_1]{2}} \rceil}) \neq 0$$

holds.

The proof of Lemma 3.2 is identical as that of Lemma 3.1, since

$$\mathcal{I}(h^m)_{x_i} = \mathcal{O}_{X,x_i} (i = 1, 2)$$

hold for every m by Proposition 2.1 and Lemma 2.1.

By Kodaira's lemma there is an effective $\mathbf{Q}$-divisor E such that $K_X - E$ is ample. Let ℓ_1 be a sufficiently large positive integer which will be specified later such that

$$L_1 := \ell_1(K_X - E)$$

is Cartier.

Lemma 3.3 *If we take ℓ_1 sufficiently large, then*

$$\phi_m : H^0(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{I}(h^m)) \to H^0(X_1, \mathcal{O}_{X_1}(mK_X + L_1) \otimes \mathcal{I}(h^m))$$

is surjective for every $m \geq 0$.

Proof. Let us take a locally free resolution of the ideal sheaf $\mathcal{I}_{X_1}$ of X_1.

$$0 \leftarrow \mathcal{I}_{X_1} \leftarrow \mathcal{E}_1 \leftarrow \mathcal{E}_2 \leftarrow \cdots \leftarrow \mathcal{E}_k \leftarrow 0.$$

Then by the trivial extension of the case of vector bundles, if ℓ_1 is sufficiently large, we see that

$$H^q(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{I}(h^m) \otimes \mathcal{E}_j) = 0$$

holds for every $m \geq 1$, $q \geq 1$ and $1 \leq j \leq k$. In fact if we take ℓ_1 sufficiently large, we see that for every j, $\mathcal{O}_X(L_1 - K_X) \otimes \mathcal{E}_j$ admits a C^∞-hermitian metric g_j such that

$$\Theta_{g_j} \geq \mathrm{Id}_{E_j} \otimes \omega$$

holds, where ω is a Kähler form on X. By [6, Theorem 4.1.2 and Lemma 4.2.2] we have the desired vanishing. Hence we have:

Sublemma 3.1

$$H^1(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{I}(h^m) \otimes \mathcal{E}_j) = 0$$

holds for every $m \geq 0$ and $1 \leq j \leq r$

Let

$$p_m : X_m \longrightarrow X$$

be a composition of successive blowing ups with smooth centers such that $p_m^*\mathcal{I}(h^m)$ is locally free on X_m.

Sublemma 3.2

$$R^p p_{m*}(\mathcal{O}_{X_m}(K_{X_m}) \otimes \mathcal{I}(p_m^* h^m)) = 0$$

holds for every $p \geq 1$ and $m \geq 1$.

We note that by the definition of multiplier ideal sheaves

$$p_{m*}(\mathcal{O}_{X_m}(K_{X_m}) \otimes \mathcal{I}(p_m^* h^m)) = \mathcal{O}(K_X) \otimes \mathcal{I}(h^m)$$

holds. Hence by Sublemma 3.1 and Sublemma 3.2 and the Leray spectral sequence, we see that

$$H^q(X_m, \mathcal{O}_{X_m}(K_{X_m} + p_m^*(mK_X + L_1 - K_X)) \otimes \mathcal{I}(p_m^* h^m) \otimes p_m^*\mathcal{E}_j) = 0$$

holds for every $q \geq 1$ and $m \geq 1$. Hence every element of

$$H^0(X_m, \mathcal{O}_{X_m}(K_{X_m} + p_m^*(mK_X + L_1 - K_X)) \otimes \mathcal{I}(p_m^* h^m) \otimes \mathcal{O}_{X_m}/p_m^*\mathcal{I}_{X_1})$$

extends to an element of

$$H^0(X_m, \mathcal{O}_{X_m}(K_{X_m} + p_m^*(mK_X + L_1 - K_X)) \otimes \mathcal{I}(p_m^* h^m)).$$

Also there exists a natural map

$$H^0(X_1, \mathcal{O}_{X_1}(mK_X + L_1) \otimes \mathcal{I}(h^m)) \to$$

$$H^0(X_m, \mathcal{O}_{X_m}(K_{X_m} + p_m^*(mK_X + L_1 - K_X)) \otimes \mathcal{I}(p_m^* h^m) \otimes \mathcal{O}_{X_m}/p_m^*\mathcal{I}_{X_1}).$$

Hence we can extend every element of

$$p_m^* H^0(X_1, \mathcal{O}_{X_1}(mK_X + L_1) \otimes \mathcal{I}(h^m))$$

to an element of

$$H^0(X_m, \mathcal{O}_{X_m}(K_{X_m} + p_m^*(mK_X + L_1 - K_X)) \otimes \mathcal{I}(p_m^* h^m)).$$

Since

$$H^0(X_m, \mathcal{O}_{X_m}(K_{X_m} + p_m^*(mK_X + L_1 - K_X)) \otimes \mathcal{I}(p_m^* h^m)) \simeq$$
$$H^0(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{I}(h^m))$$

holds by the isomorphism

$$p_{m*}(\mathcal{O}_{X_m}(K_{X_m}) \otimes \mathcal{I}(p_m^* h^m)) = \mathcal{O}(K_X) \otimes \mathcal{I}(h^m),$$

this completes the proof of Lemma 3.3. **Q.E.D.**

Let τ be a general section in $H^0(X, \mathcal{O}_X(L_1))$.

Let m_1 be a sufficiently large positive integer and let σ_1' be a general element of

$$H^0(X_1, \mathcal{O}_{X_1}(m_1 K_X) \otimes \mathcal{I}(h^{m_1}) \otimes \mathcal{M}_{x_1,x_2}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon)\frac{m_1}{\sqrt[n_1]{2}} \rceil}),$$

where $x_1, x_2 \in X_1$ are distinct nonsingular points on X_1.

By Lemma 3.2, we may assume that σ_1' is nonzero. Then by Lemma 3.3 we see that

$$\sigma_1' \otimes \tau \in H^0(X_1, \mathcal{O}_{X_1}(m_1 K_X + L_1) \otimes \mathcal{I}(h^{m_1}) \otimes \mathcal{M}_{x_1,x_2}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon)\frac{m_1}{\sqrt[n_1]{2}} \rceil})$$

extends to a section

$$\sigma_1 \in H^0(X, \mathcal{O}_X((m_1 + \ell_1)K_X) \otimes \mathcal{I}(h^{m+\ell_1}))$$

We may assume that there exists a neighbourhood $U_{x,x'}$ of $\{x, x'\}$ such that the divisor (σ_1) is smooth on $U_{x,x'} - X_1$ by Bertini's theorem, if we take ℓ_1 sufficiently large, since by Theorem 2.1,

$$H^0(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{I}(h^m)) \rightarrow H^0(X, \mathcal{O}_X(mK_X + L_1) \otimes \mathcal{I}(h^m))/\mathcal{O}_X(-X_1) \cdot \mathcal{M}_y)$$

is surjective for every $y \in X$ and $m \geq 0$, where $\mathcal{O}_X(-X_1)$ is the ideal sheaf of X_1. We define a singular hermitian metric h_1 on K_X by

$$h_1 = \frac{1}{\mid \sigma_1 \mid^{\frac{2}{m_1+\ell_1}}}.$$

Let ε_0 be a sufficiently small positive number and let $\mathcal{I}_1(\alpha)$ be the multiplier ideal sheaf of $h_0^{\alpha_0-\varepsilon_0}\cdot h_1^{\alpha}$,i.e.,

$$\mathcal{I}_1(\alpha) = \mathcal{L}^2(\mathcal{O}_X, h_0^{\alpha_0-\varepsilon_0}h_1^{\alpha}/h_X^{(\alpha_0+\alpha-\varepsilon_0)}).$$

Suppose that x, x' are nonsingular points on X_1. Then we set $x_1 = x, x_2 = x'$ and define $\alpha_1(=\alpha_1(x,y)) > 0$ by

$$\alpha_1 := \inf\{\alpha \mid (\mathcal{O}_X/\mathcal{I}_1(\alpha))_x \neq 0 \,\text{and}\, (\mathcal{O}_X/\mathcal{I}_1(\alpha))_{x'} \neq 0\}.$$

By Lemma 3.3 we may assume that we have taken m_1 so that

$$\frac{\ell_1}{m_1} \leq \varepsilon_0 \frac{\sqrt[n_1]{\mu_1}}{n_1\sqrt[n_1]{2}}$$

holds.

Lemma 3.4

$$\alpha_1 \leq \frac{n_1\sqrt[n_1]{2}}{\sqrt[n_1]{\mu_1}} + O(\varepsilon_0)$$

holds.

To prove Lemma 3.4, we need the following elementary lemma.

Lemma 3.5 *([21, p.12, Lemma 6]) Let a, b be positive numbers. Then*

$$\int_0^1 \frac{r_2^{2n_1-1}}{(r_1^2+r_2^{2a})^b}dr_2 = r_1^{\frac{2n_1}{a}-2b}\int_0^{r_1^{-2a}} \frac{r_3^{2n_1-1}}{(1+r_3^{2a})^b}dr_3$$

holds, where

$$r_3 = r_2/r_1^{1/a}.$$

Proof of Lemma 3.3. Let $(z_1,\ldots,z_n)$ be a local coordinate on a neighbourhood U of x in X such that

$$U \cap X_1 = \{q \in U \mid z_{n_1+1}(q) = \cdots = z_n(q) = 0\}.$$

We set $r_1 = (\sum_{i=n_1+1}^{n} \mid z_1 \mid^2)^{1/2}$ and $r_2 = (\sum_{i=1}^{n_1} \mid z_i \mid^2)^{1/2}$. Then there exists a positive constant C such that

$$\parallel \sigma_1 \parallel^2 \leq C(r_1^2 + r_2^{2\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon)\frac{m_1}{\sqrt[n_1]{2}}\rceil})$$

holds on a neighbourhood of x, where $\parallel \ \parallel$ denotes the norm with respect to $h_X^{m_1+\ell_1}$. We note that there exists a positive integer M such that

$$\parallel \sigma \parallel^{-2} = O(r_1^{-M})$$

holds on a neighbourhood of the generic point of $U \cap X_1$, where $\parallel \ \parallel$ denotes the norm with respect to $h_X^{m_0}$. Then by Lemma 3.5, we have the inequality

$$\alpha_1 \leq (\frac{m_1 + \ell_1}{m_1}) \frac{n_1 \sqrt[n_1]{2}}{\sqrt[n_1]{\mu_1}} + O(\varepsilon_0)$$

holds. By using the fact that

$$\frac{\ell_1}{m_1} \leq \varepsilon_0 \frac{\sqrt[n_1]{\mu_1}}{n_1 \sqrt[n_1]{2}}$$

we obtain that

$$\alpha_1 \leq \frac{n_1 \sqrt[n_1]{2}}{\sqrt[n_1]{\mu_1}} + O(\varepsilon_0)$$

holds. **Q.E.D.**

If x or x' is a singular point on X_1, we need the following lemma.

Lemma 3.6 *Let φ be a plurisubharmonic function on $\Delta^n \times \Delta$. Let $\varphi_t (t \in \Delta)$ be the restriction of φ on $\Delta^n \times \{t\}$. Assume that $e^{-\varphi_t}$ does not belong to $L^1_{loc}(\Delta^n, O)$ for every $t \in \Delta^*$.*

Then $e^{-\varphi_0}$ is not locally integrable at $O \in \Delta^n$.

Lemma 3.6 is an immediate consequence of [14]. Using Lemma 3.6 and Lemma 3.5, we see that Lemma 3.4 holds by letting $x_1 \to x$ and $x_2 \to x'$.

For the next we consider Case 1.2 and Case 2.2. We note that in Case 2.2 by modifying σ a little bit , if necessary we may assume that $(\mathcal{O}_X/\mathcal{I}(\alpha_0 - \varepsilon))_{x'} \neq 0$ and $(\mathcal{O}_X/\mathcal{I}(\alpha_0 - \varepsilon'))_x = 0$ hold for a sufficiently small positive number ε'. For example it is sufficient to replace σ by the following σ' constructed below.

Let X_1' be a subvariety which defines a branch of

$$\mathrm{Spec}(\mathcal{O}_X/\mathcal{I}(\alpha + \delta))$$

at x'. By the assumption (changing X_1, if necessary) we may assume that X_1' does not contain x. Let m' be a sufficiently large positive integer such that m'/m_0 is sufficiently small (we can take m_0 arbitrary large).

Let $\tau_{x'}$ be a general element of

$$H^0(X, \mathcal{O}_X(m'K_X) \otimes \mathcal{I}_{X_1'}),$$

where $\mathcal{I}_{X_1'}$ is the ideal sheaf of X_1'. If we take m' sufficiently large, $\tau_{x'}$ is not identically zero. We set

$$\sigma' = \sigma \cdot \tau_{x'}.$$

Then we see that the new singular hermitian metric h_0' defined by σ' satisfies the desired property.

In these cases, instead of Lemma 3.2, we use the following simpler lemma.

Lemma 3.7 *Let ε be a sufficiently small positive number and let x_1 be a smooth point on X_1. Then for a sufficiently large $m > 1$,*

$$H^0(X_1, \mathcal{O}_{X_1}(mK_X) \otimes \mathcal{I}(h^m) \otimes \mathcal{M}_{x_1}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon)m \rceil}) \neq 0$$

holds.

Then taking a general σ_1' in

$$H^0(X_1, \mathcal{O}_{X_1}(m_1 K_X) \otimes \mathcal{I}(h^{m_1}) \otimes \mathcal{M}_{x_1}^{\lceil \sqrt[n_1]{\mu_1}(1-\varepsilon)m_1 \rceil}),$$

for a sufficiently large m_1. As in Case 1.1 and Case 2.1 we obtain a proper subvariety X_2 in X_1 also in this case.

Inductively for distinct points $x, x' \in X^\circ$, we construct a strictly decreasing sequence of subvarieties

$$X = X_0 \supset X_1 \supset \cdots \supset X_r \supset X_{r+1} = \{x\} \cup R_{x'} \text{ or } \{x'\} \cup R_x,$$

where $R_{x'}$ (or R_x) is a subvariety such that x deos not belong to $R_{x'}$ and x' belongs to $R_{x'}$. and invariants (depending on small positive numbers $\varepsilon_0, \ldots, \varepsilon_{r-1}$, large positive integers $m_0, m_1, \ldots, m_r$, etc.) :

$$\alpha_0, \alpha_1, \ldots, \alpha_r,$$

$$\mu_0, \mu_1, \ldots, \mu_r$$

and

$$n > n_1 > \cdots > n_r.$$

By Nadel's vanishing theorem we have the following lemma.

Lemma 3.8 *Let x, x' be two distinct points on X°. Then for every $m \geq \lceil \sum_{i=0}^{r} \alpha_i \rceil + 1$, $\Phi_{|mK_X|}$ separates x and x'.*

Proof. Let us define the singular hermitian metric $h_{x,x'}$ of $(m-1)K_X$ defined by

$$h_{x,x'} = (\prod_{i=0}^{r-1} h_i^{\alpha_i - \varepsilon_i}) \cdot h_r^{\alpha_r + \varepsilon_r} \cdot h^{(m-1-(\sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i)) - (\alpha_r + \varepsilon_r) - \delta_L)} \cdot h_L^{\delta_L},$$

where h_L is a C^∞-hermitian metric on the $\mathbf{Q}$-line bundle $L := K_X - E$ with strictly positive curvature and δ_L be a sufficiently small positive number. Then we see that $\mathcal{I}(h_{x,x'})$ defines a subscheme of X with isolated support around x or x' by the definition of the invariants $\{\alpha_i\}$'s. By the construction the curvature current $\Theta_{h_{x,x'}}$ is strictly positive on X. Then by Nadel's vanishing theorem (Theorem 2.1) we see that

$$H^1(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h_{x,x'})) = 0.$$

This implies that $\Phi_{|mK_X|}$ separates x and x'. **Q.E.D.**

3.2 Construction of the stratification as a family

In this subsection we shall construct the above stratification as a family.

We note that for a fixed pair $(x, x') \in X^\circ \times X^\circ - \Delta_X$, $\sum_{i=0}^{r} \alpha_i$ depends on the choice of $\{X_i\}$'s, where Δ_X denotes the diagonal of $X \times X$. Moving (x, x') in $X^\circ \times X^\circ - \Delta_X$, we shall consider the above operation simultaneously. Let us explain the procedure. We set

$$B := X^\circ \times X^\circ - \Delta_X.$$

Let

$$p : X \times B \longrightarrow X$$

be the first projection and let

$$q : X \times B \longrightarrow B$$

be the second projection. Let Z be the subvariety of $X \times B$ defined by

$$Z := \{(x_1, x_2, x_3) : X \times B \mid x_1 = x_2 \text{ or } x_1 = x_3\}.$$

In this case we consider

$$q_*\mathcal{O}_{X\times B}(m_0 p^* K_X) \otimes \mathcal{I}_Z^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m_0}{\sqrt[n]{2}} \rceil}$$

instead of

$$H^0(X, \mathcal{O}_X(m_0K_X) \otimes \mathcal{M}_{x,x'}^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m_0}{\sqrt[n]{2}}\rceil}),$$

where $\mathcal{I}_Z$ denotes the ideal sheaf of Z. Let $\tilde{\sigma}_0$ be a nonzero global meromorphic section of

$$q_*\mathcal{O}_{X\times B}(m_0p^*K_X) \otimes \mathcal{I}_Z^{\lceil \sqrt[n]{\mu_0}(1-\varepsilon)\frac{m_0}{\sqrt[n]{2}}\rceil}$$

on B for a sufficiently large positive integer m_0. We define the singular hermitian metric $\tilde{h}_0$ on p^*K_X by

$$\tilde{h}_0 := \frac{1}{\mid \tilde{\sigma}_0 \mid^{2/m_0}}.$$

We shall replace α_0 by

$$\tilde{\alpha}_0 := \inf\{\alpha > 0 \mid \text{the generic point of } Z \subseteq \mathrm{Spec}(\mathcal{O}_{X\times B}/\mathcal{I}(h_0^{\alpha}))\}.$$

Then for every $0 < \delta << 1$, there exists a Zariski open subset U of B such that for every $b \in U$, $\tilde{h}_0 \mid_{X\times\{b\}}$ is well defined and

$$b \not\subseteq \mathrm{Spec}(\mathcal{O}_{X\times\{b\}}/\mathcal{I}(\tilde{h}_0^{\alpha_0-\delta} \mid_{X\times\{b\}})),$$

where we have identified b with distinct two points in X. And also by Lemma 3.6, we see that

$$b \subseteq \mathrm{Spec}(\mathcal{O}_{X\times\{b\}}/\mathcal{I}(\tilde{h}_0^{\alpha_0} \mid_{X\times\{b\}})),$$

holds for every $b \in B$. Let $\tilde{X}_1$ be an irreducible component of

$$\mathrm{Spec}(\mathcal{O}_{X\times B}/\mathcal{I}(\tilde{h}_0^{\alpha_0}))$$

containing Z. We note that $\tilde{X}_1 \cap q^{-1}(b)$ may not be irreducible even for a general $b \in B$. But if we take a suitable finite cover

$$\phi_0 : B_0 \longrightarrow B,$$

on the base change $X \times_B B_0$, $\tilde{X}_1$ defines a family of irreducible subvarieties

$$f_1 : \hat{X}_1 \longrightarrow U_0$$

of X parametrized by a nonempty Zariski open subset U_0 of $\phi_0^{-1}(U)$. We set

$$\tilde{\mu}_1 := \inf_{b_0\in U_0} \mu(f_1^{-1}(b_0), (K_X, h)).$$

We note that by its definition the volume $\mu(f_1^{-1}(b_0), (K_X, h))$ is constant on a nonempty open subset say U_0' of U_0 with respect to countable Zariski topology. We denote the constant by $\tilde{\mu}_0$. Continueing this process we may construct a finite morphism

$$\phi_r : B_r \longrightarrow B$$

and a nonempty Zariski open subset U_r of B_r which parametrizes a family of stratification

$$X \supset X_1 \supset X_2 \supset \cdots \supset X_r \supset X_{r+1} = \{x\} \cup R_{x'} (\text{resp. } \{x'\} \cup R_x)$$

constructed as before, where R_x (resp. $R_{x'}$) is a subvariety of X which is disjoint from x' (resp. x). And we also obtain invariants $\{\tilde{\alpha}_0, \ldots, \tilde{\alpha}_r\}$, $\{\tilde{\mu}_0, \ldots, \tilde{\mu}_r\}$, $\{n = \tilde{n}_0 \ldots, \tilde{n}_r\}$. Hereafter we denote these invariants without $\tilde{}$ for simplicity. By the same proof as Lemma 3.4, we have the following lemma.

Lemma 3.9

$$\alpha_i \leq \frac{n_i \sqrt[n_i]{2}}{\sqrt[n_i]{\mu_i}} + O(\varepsilon_{i-1})$$

hold for $1 \leq i \leq r$.

By Lemma 3.8 we obtain the following proposition.

Proposition 3.1 *For every*

$$m > \lceil \sum_{i=0}^{r} \alpha_i \rceil + 1$$

$\mid mK_X \mid$ *gives a birational rational map from* X *into a projective space.*

Lemma 3.10 *Let* X_i *be a strata of a very general member of the stratification parametrized by* B_r. *If* $\Phi_{|mK_X|} \mid_{X_i}$ *is birational rational map onto its image, then*

$$\deg \Phi_{|mK_X|}(X_i) \leq m^{n_i} \mu_i$$

holds.

Proof. Let $p : \tilde{X} \longrightarrow X$ be the resolution of the base locus of $\mid mK_X \mid$ and let

$$p^* \mid mK_X \mid = \mid P_m \mid + F_m$$

be the decomposition into the free part $\mid P_m \mid$ and the fixed component F_m. Let $p_i : \tilde{X}_i \longrightarrow X_i$ be the resolution of the base locus of $\Phi_{\mid mK_X \mid} \mid_{X_i}$ obtained by the restriction of p on $p^{-1}(X_i)$. Let

$$p_i^*(\mid mK_X \mid_{X_i}) = \mid P_{m,i} \mid + F_{m,i}$$

be the decomposition into the free part $\mid P_{m,i} \mid$ and the fixed part $F_{m,i}$. We have

$$\deg \Phi_{\mid mK_X \mid}(X_i) = P_{m,i}^{n_i}$$

holds. Then by the ring structure of $R(X, K_X)$, we have an injection

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\nu P_m)) \rightarrow H^0(X, \mathcal{O}_X(m\nu K_X) \otimes \mathcal{I}(h^{m\nu}))$$

for every $\nu \geq 1$, since the righthandside is isomorphic to $H^0(X, \mathcal{O}_X(m\nu K_X))$ by the definition of an AZD. We note that since $\mathcal{O}_{\tilde{X}}(\nu P_m)$ is globally generated on $\tilde{X}$, for every $\nu \geq 1$ we have the injection

$$\mathcal{O}_{\tilde{X}}(\nu P_m) \rightarrow p^*(\mathcal{O}_X(m\nu K_X) \otimes \mathcal{I}(h^{m\nu})).$$

Hence there exists a natural morphism

$$H^0(\tilde{X}_i, \mathcal{O}_{\tilde{X}_i}(\nu P_{m,i})) \rightarrow H^0(X_i, \mathcal{O}_{X_i}(m\nu K_X) \otimes \mathcal{I}(h^{m\nu})/\mathrm{tor})$$

for every $\nu \geq 1$. This morphism is clearly injective. This implies that

$$\mu_i \geq m^{-n_i} \mu(\tilde{X}_i, P_{m,i})$$

holds. Since $P_{m,i}$ is nef and big on X_i we see that

$$\mu(\tilde{X}_i, P_{m,i}) = P_{m,i}^{n_i}$$

holds. Hence

$$\mu_i \geq m^{-n_i} P_{m,i}^{n_i}$$

holds. This implies that

$$\deg \Phi_{\mid mK_X \mid}(X_i) \leq \mu_i \cdot m^{n_i}$$

holds. **Q.E.D.**

4 Proof of Theorem 1.2

Let

$$X \supset X_1 \supset X_2 \supset \cdots \supset X_r \supset X_{r+1} = \{x\} \cup R_{x'} (\text{resp. } \{x'\} \cup R_x)$$

be a very general stratification constructed as in the last section.

Suppose that $\mid \lceil (\lceil \sum_{\ell=1}^{n} \sqrt[\ell]{2}\ell \rceil + 1) K_X \mid$ does not give a birational rational map from X into a projective space. Then by Proposition 3.1, we see that

$$\max_i \frac{\alpha_i}{\sqrt[n_i]{2} n_i} \geq 1$$

holds. Let k be the number such that

$$\frac{\alpha_k}{\sqrt[n_k]{2} n_k} = \max_i \frac{\alpha_i}{\sqrt[n_i]{2} n_i}.$$

By Lemma 3.9 we see that

$$\mu_k < 1$$

holds. We set

$$\alpha := \lceil \sum_{i=0}^{r} \alpha_i \rceil + 1.$$

Now we see that by Lemma 3.10 and Lemma 3.9

$$\deg \Phi_{|\alpha K_X|}(X_k) \leq (\lceil (\sum_{\ell=1}^{n} \frac{\sqrt[\ell]{2}\ \ell}{\sqrt[n_k]{2}\ n_k}) \alpha_k \rceil + 1)^{n_k} \mu_k$$

$$\leq ((\lceil \sum_{\ell=1}^{n} \sqrt[\ell]{2}\ \ell \rceil + 1) \frac{1}{\sqrt[n_k]{\mu_k}})^{n_k} \mu_k$$

$$\leq (\lceil \sum_{\ell=1}^{n} \sqrt[\ell]{2}\ \ell \rceil + 1)^{n_k}$$

hold. Since such $\{X_k\}$ form a dominant family of subvarieties on X by the construction of the stratifications, this completes the proof of Theorem 1.2.

5 Proof of Theorem 1.3

Let X be a smooth projective n-fold of general type and suppose that $\mid (\lceil \sum_{\ell=1}^{n} \sqrt[\ell]{2}\ \ell \rceil + 1) K_X \mid$ does not give a birational embedding. Then by

Theorem 1.2 and its proof, there exists some $1 \leq d \leq n$ such that there exists a dominant family of subvarieties

$$\varpi : \mathcal{X}_k \longrightarrow S_k$$

which parametrizes the strata X_k of a general stratification

$$X \supset X_1 \supset X_2 \supset \cdots \supset X_r \supset X_{r+1} = \{x\} \cup R_{x'}(\text{resp. } \{x'\} \cup R_x)$$

such that for $\alpha := \lceil \sum_{i=0}^{r} \alpha_i \rceil + 1$

$$\deg \Phi_{|\alpha K_X|}(X_k) \leq (\lceil \sum_{\ell=1}^{n} \sqrt[\ell]{2}\ \ell \rceil + 1)^{d'}$$

holds where $d' = \dim X_k$. Let

$$p : \mathcal{X}_k \longrightarrow X$$

be the natural morphism. Inductively we define a sequence of (possibly reducible) subvarieties $F_i (i \geq 0)$ by

$$F_0 = X_k([X_k] \in S_k)$$

and for $i \geq 0$

$$F_{i+1} = \text{the closure of } \ p(\varpi^{-1}(\varpi(\pi^{-1}(\text{the generic points of } F_i)))).$$

Then $\{F_i\}_{i \geq 0}$ is increasing and by the Noetherian property, we see that there exists some $\ell \geq 0$ such that

$$\dim F_\ell = \dim F_{\ell'}$$

for every $\ell' \geq \ell$. Let $F_{\ell,0}$ be a maximal dimensional component of F_ℓ.

If we start from a general $[X_k] \in S_k$ and choose $F_{\ell,0}$ properly, we may assume that $\{F_{\ell,0}\}$ form a family. We note that possibly $F_{\ell,0}$ may not be determined only by X_k because of a monodoromy phenomenon. Let

$$\varpi_0 : \mathcal{U}_0 \longrightarrow T_0$$

be the family of such $\{F_{\ell,0}\}$. Then it is again dominant. Let

$$p_0 : \mathcal{U}_0 \longrightarrow X$$

be the natural morphism. We see that p_0 is birational, since for a general $x \in X$, $p_0^{-1}(x)$ is a point (otherwise it contradicts to the maximality of $\dim F_{\ell,0}$). Hence ϖ_0 induces a rational fibration structure

$$f : X - \cdots \to Y.$$

Let F be a general fiber of f. To completes the proof of Theorem 1.3 we need the following lemma.

Lemma 5.1

$$\deg \Phi_{|\alpha K_X|}(F) \leq (\lceil \sum_{\ell=1}^{n} \sqrt[\ell]{2}\ \ell \rceil + 1)^{d^2} d^d$$

Proof. By taking a suitable modification of X, we may assume that the following conditions are satisfied :

1. $\Phi_{|\alpha K_X|}$ is a morphism on X,
2. there exists a regular fibration

   $$f : X \longrightarrow Y$$

   induced by $p_0 : \mathcal{U}_0 \longrightarrow X$ as above.

Let $\mid H \mid$ be the free part of $\mid \alpha K_X \mid$. We set

$$a := (\lceil \sum_{\ell=1}^{n} \sqrt[\ell]{2}\ \ell \rceil + 1)^{d'}.$$

Let F be a general fiber of f. Suppose that

$$\deg \Phi_{|\alpha K_X|}(F) > a^d d^d$$

holds.

Lemma 5.2 *Let us fix an arbitrary point x_0 on F. Then for a sufficiently large m there exists a section*

$$\sigma \in \Gamma(F, \mathcal{O}_F(mH)) - \{0\}$$

such that

$$mult_{x_0}(\sigma) > mad + 1$$

holds.

Proof. Since

$$\dim \mathcal{O}_F/\mathcal{M}_{x_0}^m = \left(\begin{array}{c} d+m-1 \\ d \end{array} \right) = \frac{1}{d!}m^d + O(m^{d-1})$$

and

$$\dim H^0(F, \mathcal{O}_X(mH)) = \frac{1}{d!}a^d d^d m^d + O(m^{d-1})$$

hold, the lemma is clear. **Q.E.D.**

Let

$$\varpi_F : \mathcal{X}_k(F) \longrightarrow S_k(F)$$

be the family of the strata X_k contained in F. Then there exists a subvariety $S'_k(F)$ such that

1. $\dim S'_k(F) = \dim F - \dim X_k$,
2. We set $\tilde{F} := \varpi_F^{-1}(S'_k(F))$. Then $p_{\tilde{F}} : \tilde{F} \longrightarrow F$ is generically finite.

Then

$$\tilde{\varpi}_F : \tilde{F} \longrightarrow S'_k(F)$$

is an algebraic fiber space. We set

$$W_j = \{\tilde{x} \in \tilde{F} \mid \mathrm{mult}_{\tilde{x}}\ p_{\tilde{F}}^*(\sigma) \geq 1 + maj\}.$$

We have a decending chain of subvarieties

$$W_0 \supset W_1 \supset \cdots \supset W_d \ni p_{\tilde{F}}^{-1}(x_0).$$

For each $j \in \{0, \ldots, n\}$, we choose an irreducible component of W_j containing x, and denote this irreducible component by W'_j. We may assume that these irreducible components have been chosen so that

$$W'_0 \supset W'_1 \supset \cdots \supset W'_d \ni p_{\tilde{F}}^{-1}(x_0).$$

Since this chain has length greater than $d = \dim F$, there exists some j such that

$$W'_j = W'_{j+1}$$

holds. We set $W = W'_j$.

Let $\mathcal{C}$ denote the family of irreducible curves

$$C = H_1 \cap \cdots \cap H_{d'-1} \cap X'_k$$

on F which is obtained as the intersection of $(d'-1)$-members $H_1, \ldots H_{d'-1}$ of $\mid H \mid$ and a strata $X'_k([X'_k] \in S_k)$ which is contained in F. We note that for a general member C of $\mathcal{C}$, the inverse image of C in the normalization of X'_k is smooth. Hence a general member of $\mathcal{C}$ is immersed in X that means the differential of the natural morphism from the normalization of C to X is nowhere vanishing. By the construction $\mathcal{C}$ determines a dominant family of curves $\tilde{\mathcal{C}}$ on $\tilde{F}$. We note that by the construction of F a member X'_k of S_k intersects F, then X'_k should be contained in F. Now we note have that

$$H \cdot C \leq a$$

holds.

Now we quote the following two theorems (the satements are slightly generalized, but the proofs are completely same).

Theorem 5.1 *([13, p. 686,Theorem 2]) Fix a positve integers ℓ and k. Let $f : M \longrightarrow \Delta^k$ be a smooth family of irreducible curves. Let L be a holomorphic line bundle on M such that the restriction of L to $f^{-1}(0)$ has degree ℓ and let $s \in H^0(M, \mathcal{O}_M(L))$. Let $V_j(s)$ denote the complex subspace of M consinsting precisely those points at which the vanishing order of s is at least j.*

Then either

$$f^{-1}(0) \cap V_{j+\ell}(s) = \emptyset$$

or

$$f^{-1}(0) \subset V_j(s)$$

holds.

Theorem 5.2 *([13, p.686,Theorem 3]) Let M be a complex manifold, let L be a holomorphic line bundle on M and let $s \in H^0(M, \mathcal{O}_M(L))$. Let $\mathcal{N}$ be an irreducible family of immersed curves thich is free and set $\ell := \deg_N L$, where N is a member of $\mathcal{N}$. For any integer j, either*

$$N \cap V_{j+\ell}(s) = \emptyset$$

or

$$N \subset V_j(s)$$

holds.

Lemma 5.3 *If $\tilde{C}$ be an immersed member of $\tilde{\mathcal{C}}$ which intersects W_j, then $\tilde{C}$ is contained in W_{j-1}.*

Proof. We note that $\tilde{C}$ is free by the construction. Then the lemma follows from Theorem 5.2. **Q.E.D.**

Now we fix a general W. Then by the definition of $\mathcal{C}$ and Lemma 5.1 implies that if for a fiber $\tilde{X}_k$ of

$$\tilde{\varpi}_F : \tilde{F} \longrightarrow S'_k(F)$$

$$\tilde{X}_k \cap W \neq \emptyset$$

holds, then

$$\tilde{X}_k \subseteq W$$

holds. We note that W is defined by the pullback of the section of mH on F. By the inductive construction of F, we see that if we take W general, $W = \tilde{F}$ holds. This is the contradiction. Hence we see that

$$\deg \Phi_{|\alpha K_X|}(F) \leq a^d d^d$$

holds. Since

$$a^d d^d \leq (\lceil \sum_{\ell=1}^{n} \sqrt[\ell]{2}\ \ell \rceil + 1)^{d^2} d^d$$

holds, this completes the proof of Theorem 1.3.

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